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SYMMETRIES OF INTEGRABLE DEFORMATIONS

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ABSTRACT. In this note , we will explain symmetries of isomonodromic deformations as Weyl groups of some quivers and give classifications of isomondromic deformations of linear ordinary differential equations with at most unramified irregular singularities and 2 or 4 accessory parameters.

INTRODUCTION

In the series of works by Okamoto [30], it was clarified that Painlevé equations have affine Weyl group symmetries. After these pioneering works, many studies of symmetries of Painlevé type equations are successfully developed in connection with the algebraic geometry, representation theory of affine Lie algebras and so on (see Noumi and Yamada [29], Sakai [32], Sasano [34], Boalch [5] and their references for instance). On the other hand, the recent work of Kawakami, Nakamura and Sakai [23] suggests that many known Painlevé type equations are uniformly obtained from isomonodormic deformations of linear ordinary differential equations. In this note, inspired by their work, we shall introduce a study of symmetries of isomonodromic deformations from those of moduli spaces of meromorphic connections. The detail of this note can be found in the paper [13]

Let us explain the organization of this note. The first section is a preliminary which collects necessary notions for the latter sections, gauge transformations of differential equations, Hukuhara-Turrittin normal forms, and quiver varieties. In the second section, we explain a realization of the moduli space of differential equations as quiver variety. In the final section, we will discuss relationship among middle convolution on differential equations, the Weyl group action on quiver varieties, and the symmetry of isomonodromic deformations. Also we shall apply a classification of root systems to that of isomonodromic deformations.

1. PRELIMINARIES

For a commutative ring R, $M(n,R)$ denotes the set of $n\times n$ matrices with coefficients in R and $\mathrm{GL}(n,R)\subset M(n,R)$ consists of invertible elements. The sheaves of holomorphic functions and meromorphic functions

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on a complex manifold X are written by $\mathcal{O}_X$ and $\mathcal{M}_X$ respectively. In particular when $X = \mathbb{P}^1$, we write $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{M} = \mathcal{M}_{\mathbb{P}^1}$ for short. Let us denote the ring of convergent (resp. formal) power series of z by $\mathbb{C}\{z\}$ (resp. $\mathbb{C}[\![z]\!]$). Their total quotient fields are written by $\mathbb{C}\{\{z\}\}$ and $\mathbb{C}(\!(z)\!)$ respectively.

1.1. **Gauge equivalences of differential equations.** We recall gauge transformations of systems of first order linear ordinary differential equations defined locally on $\mathbb{P}^1$ and moreover recall Hukuhara-Turrittin-Levelt normal forms of local differential equations under formal gauge transformations.

Let U be an open subset of $\mathbb{P}^1$ and z a local coordinate on U.

Definition 1.1 (gauge transformation). For a linear differential equation

$$\frac{d}{dz}Y = AY$$

with $A \in M(n, \mathcal{M}(U))$ and $X \in \mathrm{GL}(n, \mathcal{M}(U))$, we define a new differential equation $\frac{d}{dz}\tilde{Y} = B\tilde{Y}$ by

$$B := XAX^{-1} + \left(\frac{d}{dz}X\right)X^{-1}.$$

We call B the *meromorphic gauge transformation* of A by X and write $B =: X[A]$. In particular if $X \in \mathrm{GL}(n, \mathcal{O}(U))$, we say the *holomorhic gauge transformation.*

Here we note that if a vector Y is a solution of $\frac{d}{dz}Y = AY$ then $\tilde{Y} = XY$ is a solution of $\frac{d}{dz}\tilde{Y} = B\tilde{Y}$ for $B = X[A]$.

Let us take $a \in U$ and choose a local coordinate z which is zero at a. Then the stalks $\mathcal{O}_a$ and $\mathcal{M}_a$ at a can be identified with $\mathbb{C}\{z\}$ and $\mathbb{C}\{\{z\}\}$. We can similarly define holomorphic and meromorphic gauge transformations of a local differential equation $\frac{d}{dz}Y = AY$ with $A \in M(n, \mathcal{M}_a)$. In this case, we can moreover define formal gauge transformations, namely we say $X[A]$ is the *formal holomorphic gauge transformation* of A by X if $X \in \mathrm{GL}(n, \mathbb{C}[\![z]\!])$ and *formal meromorphic gauge transformation* if $X \in \mathrm{GL}(n, \mathbb{C}(\!(z)\!))$.

For a local differential equation $\frac{d}{dz}Y = AY$ with $A \in M(n, \mathbb{C}(\!(z)\!))$, it is known that there exists a normal form under the formal meromorphic gauge transformations as follows.

Let $\mathcal{P} := \bigcup_{s\in\mathbb{Z}_{>0}} \mathbb{C}(\!(z^{\frac{1}{s}})\!)$, the field of Puiseux series.

Definition 1.2 (Hukuhara-Turrittin-Levelt normal form). By *Hukuhara-Turrittin-Levelt normal form* or *HTL normal form* for short, we mean an element of the form

$$\mathrm{diag}\left(q_1(z^{-\frac{1}{s}})I_{n_1} + R_1, \ldots, q_m(z^{-\frac{1}{s}})I_{n_m} + R_m\right) z^{-1}$$
$$\in M(n, \mathbb{C}(\!(z^{\frac{1}{s}})\!)) \subset M(n, \mathcal{P})$$

where $q_i(t) \in t\mathbb{C}[t]$ satisfying $q_i \neq q_j$ if $i \neq j$, and $R_i \in M(n_i, \mathbb{C})$ with $n_1 + \cdots n_m = n$.

For an HTL normal form $H \in M(n, \mathcal{P})$, we call $H_{\mathrm{irr}} := H - \mathrm{pr}_{\mathrm{res}}(H)z^{-1}$ the *irregular part* of H. Here we denote the coefficient matrix of z^{-1} in $A \in M(n, \mathcal{P})$ by $\mathrm{pr}_{\mathrm{res}}(A)$.

The following is a fundamental fact of the local formal theory of differential equations with irregular singularity.

Theorem 1.3 (Hukuhara-Turrittin-Levelt, see [36] for instance). *For any $A \in M(n, \mathbb{C}((z)))$, there exist an integer $r \in \mathbb{Z}_{>0}$ and $X \in \mathrm{GL}(n, \mathbb{C}((z^{\frac{1}{r}})))$ such that $X[A]$ is an HTL normal form in $M(n, \mathbb{C}((z^{\frac{1}{r}})))$. We call this $X[A]$ the* normal form *of A.*

In the above theorem, we may assume the field extension is minimal, i.e.,

$$r = \min\left\{ s \,\middle|\, \text{normal form } X[A] \in M(n, \mathbb{C}((z^{\frac{1}{s}}))) \right\}.$$

If two HTL normal forms $H, H' \in M(n, \mathbb{C}((z^{\frac{1}{r}})))$ are normal forms of an $A \in M(n, \mathbb{C}((z)))$, then there exists $g \in \mathrm{GL}(n, \mathbb{C})$ such that

$$g^{-1} H_{\mathrm{irr}} g = H'_{\mathrm{irr}}, \quad g^{-1} \exp(2\pi\sqrt{-1}k \, \mathrm{pr}_{\mathrm{res}}(H)) g = \exp(2\pi\sqrt{-1}k \, \mathrm{pr}_{\mathrm{res}}(H'))$$

for some integer $k \geq 1$, see Theorem 6.3 in [2] for example.

1.2. **Quiver varieties.** In this subsection we shall introduce quiver varieties.

1.2.1. *Representations of quiver and quiver variety.* Now let us recall representations of quivers

Definition 1.4 (quiver). A *quiver* $\mathsf{Q} = (\mathsf{Q}_0, \mathsf{Q}_1, s, t)$ is the quadruple consisting of Q_0, the set of *vertices*, and Q_1, the set of *arrows* connecting vertices in Q_0, and two maps $s, t : \mathsf{Q}_1 \to \mathsf{Q}_0$, which associate to each arrow $\rho \in \mathsf{Q}_1$ its *source* $s(\rho) \in \mathsf{Q}_0$ and its *target* $t(\rho) \in \mathsf{Q}_0$ respectively.

For a fixed vector $\alpha \in (\mathbb{Z}_{\geq 0})^{\mathsf{Q}_0}$, the representation space of the quiver with the dimension vector α is

$$\mathrm{Rep}\,(\mathsf{Q}, \alpha) = \bigoplus_{\rho \in \mathsf{Q}_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho)}}, \mathbb{C}^{\alpha_{t(\rho)}}).$$

Let us recall the double of a quiver Q.

Definition 1.5 (double quiver). Let $\mathsf{Q} = (\mathsf{Q}_0, \mathsf{Q}_1)$ be a finite quiver. Then the *double quiver* $\overline{\mathsf{Q}}$ of Q is the quiver obtained by adjoining the reverse arrow $\rho^* \colon b \to a$ to each arrow $\rho \colon a \to b$. Namely $\overline{\mathsf{Q}} := (\overline{\mathsf{Q}}_0 := \mathsf{Q}_0, \overline{\mathsf{Q}}_1 := \mathsf{Q}_1 \cup \mathsf{Q}_1^*)$ where $\mathsf{Q}_1^* := \{\rho^* \colon t(\rho) \to s(\rho) \mid \rho \in \mathsf{Q}_1\}$.

Let us note that for each $\rho \in \mathsf{Q}_1$ we can identify

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho)}}, \mathbb{C}^{\alpha_{t(\rho)}})^* \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho^*)}}, \mathbb{C}^{\alpha_{t(\rho^*)}})$$

by the trace pairing. Thus the representation space $\mathrm{Rep}(\overline{\mathsf{Q}}, \alpha)$ can be identified with the cotangent bundle

$$T^*\mathrm{Rep}(\mathsf{Q}, \alpha) \cong \mathrm{Rep}(\overline{\mathsf{Q}}, \alpha).$$

In this case the canonical symplectic form is given by

$$\omega(x, y) = \sum_{\rho \in \mathsf{Q}_1} (\mathrm{tr}(x_\rho y_{\rho^*}) - \mathrm{tr}(x_{\rho^*} y_\rho)).$$

Thus we can see $\mathrm{Rep}(\overline{\mathrm{Q}},\alpha)$ as a complex symplectic manifold with the action of

$$\mathbf{G} := \prod_{a\in \mathrm{Q}_0} \mathrm{GL}(\alpha_a,\mathbb{C}).$$

Then the following map is a moment map;

$$\mu_\alpha\colon \mathrm{Rep}(\overline{\mathrm{Q}},\alpha) \to \prod_{a\in\mathrm{Q}_0} M(\alpha_a,\mathbb{C})$$

whose images $(\mu_\alpha(x)_a)_{a\in\mathrm{Q}_0}$ are given by

$$\mu_\alpha(x)_a = \sum_{\substack{\rho\in\mathrm{Q}_1\\ t(\rho)=a}} x_\rho x_{\rho^*} - \sum_{\substack{\rho\in\mathrm{Q}_1\\ s(\rho)=a}} x_{\rho^*}x_\rho.$$

Now we are ready to define quiver varieties.

Definition 1.6 (quiver variety). Let us take a collection of complex numbers $\lambda = (\lambda_a) \in \mathbb{C}^{\mathrm{Q}_0}$. Then a *quiver variety* is the affine quotient

$$\mathfrak{M}_\lambda(\mathrm{Q},\alpha) := \mu^{-1}(\lambda)/\!/\mathbf{G} := \mathrm{Specm}\,\mathbb{C}[\mu^{-1}(\lambda)]^{\mathbf{G}}.$$

Here $\mathbb{C}[\mu^{-1}(\lambda)]$ is the coordinate ring of $\mu^{-1}(\lambda)$. Let us consider the (possibly empty) subspace

$$\mu^{-1}(\lambda)^{\mathrm{irr}} := \{x\in\mu^{-1}(\lambda) \mid x \text{ is irreducible}\}.$$

Then the action of $\mathbf{G}/\mathbb{C}^\times$ on this space is proper and moreover free (see King [24]). Thus the symplectic reduction

$$\mathfrak{M}^{\mathrm{reg}}_\lambda(\mathrm{Q},\alpha) := \mu^{-1}(\lambda)^{\mathrm{irr}}/\mathbf{G}$$

can be seen as a complex manifold with the symplectic structure, i.e., a complex symplectic manifold. We call this manifold the quiver variety too.

Remark 1.7. The above quiver varieties are special ones of Nakajima quiver varieties which enjoy rich geometric properties and applications for representation theory and theoretical physics and so on (see [27] for instance).

1.2.2. *Some geometry of quiver varieties.* As we noted before, the complex symplectic manifold $\mathfrak{M}^{\mathrm{reg}}_\lambda(\mathrm{Q},\alpha)$ is possibly empty. Thus next we see a necessary and sufficient condition for the non-emptiness of $\mathfrak{M}^{\mathrm{reg}}_\lambda(\mathrm{Q},\alpha)$ obtained by Crawley-Boevey in [6].

In order to explain the condition, recall the root system of a quiver Q (cf. [19]). Let Q be a finite quiver. From the *Euler form*

$$\langle\alpha,\beta\rangle := \sum_{a\in\mathrm{Q}_0}\alpha_a\beta_a - \sum_{\rho\in\mathrm{Q}_1}\alpha_{s(\rho)}\beta_{t(\rho)},$$

a symmetric bilinear form and quadratic form are defined by

$$(\alpha,\beta) := \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle,$$
$$q(\alpha) := \frac{1}{2}(\alpha,\alpha)$$

and set $p(\alpha) := 1 - q(\alpha)$. Here $\alpha,\beta\in\mathbb{Z}^{\mathrm{Q}_0}$.

For each vertex $a\in\mathrm{Q}_0$, define $\epsilon_a\in\mathbb{Z}^{\mathrm{Q}_0}$ $(a\in\mathrm{Q}_0)$ so that $(\epsilon_a)_a = 1$, $(\epsilon_a)_b = 0$, $(b\in\mathrm{Q}_0\backslash\{a\})$. We call ϵ_a a *fundamental root* if the vertex a has no edge-loop, i.e., there is no arrow ρ such that $s(\rho) = t(\rho) = a$. Denote by Π the set of fundamental roots. For a fundamental root ϵ_a, define the *fundamental reflection* s_a by

$$s_a(\alpha) := \alpha - (\alpha, \epsilon_a)\epsilon_a \text{ for } \alpha \in \mathbb{Z}^{Q_0}.$$

The group $W \subset \operatorname{Aut} \mathbb{Z}^{Q_0}$ generated by all fundamental reflections is called the *Weyl group* of the quiver Q. Note that the bilinear form $(\,,\,)$ is W-invariant. Similarly we can define the reflection $r_a \colon \mathbb{C}^{Q_0} \to \mathbb{C}^{Q_0}$ by

$$r_a(\lambda)_b := \lambda_b - (\epsilon_a, \epsilon_b)\lambda_a$$

for $\lambda \in \mathbb{C}^{Q_0}$ and $a, b \in Q_0$. Define the set of *real roots* by

$$\Delta^{\mathrm{re}} := \bigcup_{w \in W} w(\Pi).$$

Define the *fundamental set* $F \subset \mathbb{Z}^{Q_0}$ by

$$F := \left\{ \alpha \in (\mathbb{Z}_{\geq 0})^{Q_0} \backslash \{0\} \mid (\alpha, \epsilon) \leq 0 \text{ for all } \epsilon \in \Pi, \text{ support of } \alpha \text{ is connected} \right\}.$$

Then define the set of *imaginary roots* by

$$\Delta^{\mathrm{im}} := \bigcup_{w \in W} w(F \cup -F).$$

Then the *root system* is

$$\Delta := \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}.$$

An element $\Delta^+ := \alpha \in \Delta \cap (\mathbb{Z}_{\geq 0})^{Q_0}$ is called a *positive root.*

Now we are ready to see Crawley-Boevey's theorem. For a fixed $\lambda = (\lambda_a) \in \mathbb{C}^{Q_0}$, the set Σ_λ consists of the positive roots satisfying

(1) $\lambda \cdot \alpha := \sum_{a \in Q_0} \lambda_a \alpha_a = 0$,
(2) if there exists a decomposition $\alpha = \beta_1 + \beta_2 + \cdots$, with $\beta_i \in \Delta^+$ and $\lambda \cdot \beta_i = 0$, then $p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots$.

Theorem 1.8 (Crawley-Boevey. Theorem 1.2 in [6]). *Let Q be a finite quiver and $\overline{Q}$ the double of Q. Let us fix a dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$ and $\lambda \in \mathbb{C}^{Q_0}$. Then $\mu^{-1}(\lambda)^{irr} \subset \operatorname{Rep}(\overline{Q}, \alpha)$ is nonempty if and only if $\alpha \in \Sigma_\lambda$. Furthermore, in this case $\mu^{-1}(\lambda)$ is an irreducible algebraic variety and $\mu^{-1}(\lambda)^{irr}$ is dense in $\mu^{-1}(\lambda)$.*

Moreover Crawley-Boevey showed the following geometric properties of quiver varieties.

Theorem 1.9 (Crawley-Boevey Corollary 1.4 in [6]). *If $\alpha \in \Sigma_\lambda$ then the quiver variety $\mathfrak{M}_\lambda(Q, \alpha)$ is a reduced and irreducible variety of dimension $2p(\alpha)$.*

Combining these results, we have the following non-emptiness condition of regular parts of quiver varieties.

Corollary 1.10 (Crawley-Boevey [6]). *The quiver variety $\mathfrak{M}_\lambda^{reg}(Q, \alpha)$ is non-empty if and only if $\alpha \in \Sigma_\lambda$. Furthermore in this case, it is a connected complex symplectic manifold of dimension $2p(\alpha)$.*

2. Moduli spaces of stable meromorphic connections on trivial bundles

Let us define moduli spaces of meromorphic connections on trivial bundles following Boalch's paper [3] (see also [17]).

2.1. Moduli spaces of meromorphic connections. Let

$$B = \operatorname{diag}(q_1(z^{-1})I_{n_1} + R_1 z^{-1}, \cdots q_m(z^{-1})I_{n_m} + R_m z^{-1}) \in \mathrm{GL}(n, \mathbb{C}((z)))$$

be an HTL normal form. The equivalent class of B under formal holomorphic gauge transformations is

$$\mathcal{O}_B := \{X[B] \mid X \in \mathrm{GL}(n, \mathbb{C}[\![z]\!])\}.$$

Let us consider another equivalent class of B called the *truncated orbit* of B. Let us consider the projection

$$\iota \colon M(n, \mathbb{C}((z))) \longrightarrow M(n, \mathbb{C}((z))/\mathbb{C}[\![z]\!]).$$

The map ι induces the action of $\mathrm{GL}(n, \mathbb{C}[\![z]\!])$ on $M(n, \mathbb{C}((z))/\mathbb{C}[\![z]\!])$ from the adjoint action of that on $M(n, \mathbb{C}((z)))$. Namely for $g \in \mathrm{GL}(n, \mathbb{C}[\![z]\!])$, $Z \in M(n, \mathbb{C}((z))/\mathbb{C}[\![z]\!])$ define $g^{-1}Zg := \iota(g\tilde{Z}g^{-1})$ where $\tilde{Z} \in M(n, \mathbb{C}((z)))$ is chosen so that $\iota(\tilde{Z}) = Z$. We can see that this is independent of the choice of $\tilde{Z}$.

Then regarding B as an element in $M(n, \mathbb{C}((z))/\mathbb{C}[\![z]\!])$, we define the truncated orbit of B by the action of $\mathrm{GL}(n, \mathbb{C}[\![z]\!])$ on $M(n, \mathbb{C}((z))/\mathbb{C}[\![z]\!])$;

$$\mathcal{O}_B^{\mathrm{tru}} := \left\{g^{-1}Bg \in M(n, \mathbb{C}((z))/\mathbb{C}[\![z]\!]) \,\middle|\, g \in \mathrm{GL}(n, \mathbb{C}[\![z]\!])\right\}.$$

Let us consider a meromorphic connection $(\mathcal{O}^n, \nabla)$ on the trivial bundle over $\mathbb{P}^1$. We write $\nabla_a \in \mathcal{O}_B$ (resp. $\mathcal{O}_B^{\mathrm{tru}}$) for $a \in \mathbb{P}^1$ if there exists $A_a \in M(n, \mathbb{C}((z_a)))$ such that $\nabla = d - A_a\, dz_a$ near a and $A_a \in \mathcal{O}_B$ (resp. $\iota(A_a) \in \mathcal{O}_B^{\mathrm{tru}}$). Here $z_a := \begin{cases} z - a & \text{if } a \in \mathbb{C} \\ 1/z & \text{if } a = \infty \end{cases}$ with the standard coordinate z of $\mathbb{C}$. Let $S = k_0a_0 + \ldots + k_pa_p$ be an effective divisor on $\mathbb{P}^1$ as before. Define a set of meromorphic connections on the trivial bundle $\mathcal{O}^n$ over $\mathbb{P}^1$

$$\mathrm{Triv}_S^{(n)} := \left\{(\mathcal{O}^n, \nabla) \middle| \ \nabla \colon \mathcal{O}^n \to \mathcal{O}^n \otimes \Omega_S \ \right\}.$$

We say $(\mathcal{O}^n, \nabla) \in \mathrm{Triv}_S^{(n)}$ is *stable* if there exists no nontrivial proper subspace $W \subset \mathbb{C}^n$ such that the subbundle $\mathcal{W} := W \otimes \mathcal{O} \subset \mathbb{C}^n \otimes \mathcal{O} = \mathcal{O}^n$ is closed under ∇, i.e.,

$$\nabla(\mathcal{W}) \subset \mathcal{W} \otimes \Omega_S.$$

Let $\mathbf{B} = (B_0, \ldots, B_p) \in M(n, \mathbb{C}((z)))^{p+1}$ be a collection of HTL normal forms satisfying $\operatorname{ord}(B_i) = k_i$ for all $i = 0, \ldots, p$. Then the moduli space of stable meromorphic connections on trivial bundles is

$$\mathfrak{M}(\mathbf{B}) := \left\{(\mathcal{O}^n, \nabla) \in \mathrm{Triv}_S^{(n)} \,\middle|\, \begin{array}{c} (\mathcal{O}^n, \nabla)\colon \text{ stable}, \\ \nabla_{a_i} \in \mathcal{O}_{B_i}^{\mathrm{tru}} \text{ for all } i = 0, \ldots, p \end{array} \right\} / \mathrm{GL}(n, \mathbb{C}).$$

Here $\mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathcal{O}(\mathbb{P}^1))$ acts on $\mathrm{Triv}_S^{(n)}$ as holomorphic gauge transformations.

We can identify meromorphic connections on trivial bundles over $\mathbb{P}^1$ and linear ordinary differential equations on $\mathbb{P}^1$. Thus we can regard $\mathfrak{M}(\mathbf{B})$ as a moduli space of meromorphic differential equations on $\mathbb{P}^1$,

$$\mathfrak{M}(\mathbf{B}) =$$

$$\left\{ \frac{d}{dz}Y = \left(\sum_{i=1}^{p}\sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{(z-a_i)^\nu} - \sum_{2\le\nu\le k_0} A_\nu^{(0)} z^{\nu-2} \right) Y \;\middle|\; \begin{array}{c} \text{irreducible,} \\ \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{z^\nu} \in \mathcal{O}_{B_i}^{\mathrm{tru}}, \\ i = 0,\dots,p \end{array} \right\} \Big/ \mathrm{GL}(n,\mathbb{C}).$$

Here we set

$$A_1^{(0)} := -\sum_{i=1}^{p} A_1^{(i)}.$$

2.2. **Moduli spaces of connections and quiver varieties.** We shall give a realization of the moduli space $\mathfrak{M}(\mathbf{B})$ as a quiver variety. Let us suppose that $B^{(0)},\dots,B^{(p)}$ are written by

$$B^{(i)} = \mathrm{diag}\left(q_1^{(i)}(z^{-1})I_{n_1^{(i)}} + R_1^{(i)}z^{-1},\dots,q_{m^{(i)}}^{(i)}(z^{-1})I_{n_{m^{(i)}}^{(i)}} + R_{m^{(i)}}^{(i)}z^{-1} \right)$$

and choose complex numbers $\xi_1^{[i,j]},\dots,\xi_{e_{[i,j]}}^{[i,j]}$ so that

$$\prod_{k=1}^{e_{[i,j]}} (R_j^{(i)} - \xi_k^{[i,j]}) = 0$$

for $i = 0,\dots,p$ and $j = 1,\dots,m^{(i)}$. Set

$$k_i := -\max_{j=1,\dots,m^{(i)}} \{\mathrm{ord}(q_j^{(i)}(z^{-1}))\}$$

for each $i = 0,\dots,p$. Set

$$I_{\mathrm{irr}} := \{ i \in \{0,\dots,p\} \mid m^{(i)} > 1 \} \cup \{0\}$$

and

$$I_{\mathrm{reg}} := \{0,\dots,p\} \backslash I_{\mathrm{irr}}.$$

Here I_{irr} may be seen as the set of irregular singular points and ∞, and I_{reg} of regular singular points other than ∞.

Then let us define a quiver Q as follows. Set

$$\mathsf{Q}_0^{\mathrm{irr}} := \left\{ [i,j] \;\middle|\; \begin{array}{l} i \in I_{\mathrm{irr}}, \\ j = 1,\dots,m^{(i)} \end{array} \right\}, \quad \mathsf{Q}_0^{\mathrm{leg}} := \left\{ [i,j,k] \;\middle|\; \begin{array}{l} i = 0,\dots,p, \\ j = 1,\dots,m^{(i)}, \\ k = 1,\dots,e_{[i,j]} - 1 \end{array} \right\}.$$

Then the set of vertices of Q is the disjoint union

$$\mathsf{Q}_0 := \mathsf{Q}_0^{\mathrm{irr}} \sqcup \mathsf{Q}_0^{\mathrm{leg}}.$$

Also set

$$\begin{aligned}
\mathsf{Q}_1^{0\to I_{\mathrm{irr}}} &:= \left\{ \rho_{[i,j']}^{[0,j]}\colon [0,j]\to[i,j'] \;\middle|\; \begin{array}{l} j=1,\dots,m^{(0)},\\ i\in I_{\mathrm{irr}}\backslash\{0\},\\ j'=1,\dots,m^{(i)} \end{array} \right\},\\
\mathsf{Q}_1^{B^{(i)}} &:= \left\{ \rho_{[i,j],[i,j']}^{[k]}\colon [i,j]\to[i,j'] \;\middle|\; \begin{array}{l} 1\le j<j'\le m^{(i)},\\ 1\le k\le d_i(j,j') \end{array} \right\},\\
\mathsf{Q}_1^{\mathrm{leg}^{(i)}} &:= \left\{ \rho_{[i,j,k]}\colon [i,j,k]\to[i,j,k-1] \;\middle|\; \begin{array}{l} j=1,\dots,m^{(i)},\\ k=2,\dots,e_{[i,j]}-1 \end{array} \right\},\\
\mathsf{Q}_1^{\mathrm{leg}^{(i)}\to B^{(i)}} &:= \left\{ \rho_{[i,j,1]}\colon [i,j,1]\to[i,j] \mid j=1,\dots,m^{(i)} \right\},\\
\mathsf{Q}_1^{\mathrm{leg}^{(i)}\to 0} &:= \left\{ \rho_{[0,j]}^{[i,1,1]}\colon [i,1,1]\to[0,j] \mid i\in I_{\mathrm{reg}},\ j=1,\dots,m^{(0)} \right\}.
\end{aligned}$$

Here $d_i(j,j') := \deg_{\mathbb{C}[z]}(q_j^{(i)}(z)-q_{j'}^{(i)}(z))-2$.

Then the set of arrows of Q is the disjoint union

$$\mathsf{Q}_1 := \mathsf{Q}_1^{0\to I_{\mathrm{irr}}} \sqcup \bigsqcup_{i\in I_{\mathrm{irr}}} \left(\mathsf{Q}_1^{B^{(i)}} \sqcup \mathsf{Q}_1^{\mathrm{leg}^{(i)}\to B^{(i)}} \sqcup \mathsf{Q}_1^{\mathrm{leg}^{(i)}} \right) \sqcup \bigsqcup_{i\in I_{\mathrm{reg}}} \left(\mathsf{Q}_1^{\mathrm{leg}^{(i)}\to 0} \sqcup \mathsf{Q}_1^{\mathrm{leg}^{(i)}} \right).$$

Let $\alpha = (\alpha_a)_{a\in\mathsf{Q}_0} \in \mathbb{Z}^{\mathsf{Q}_0}$ be the vector,

$$\alpha_{[i,j]} := n_j^{(i)} \quad \text{and} \quad \alpha_{[i,j,k]} := \operatorname{rank} \prod_{l=1}^{k} (R_j^{(i)} - \xi_l^{[i,j]}).$$

Also define $\lambda = (\lambda_a)_{a\in\mathsf{Q}_0} \in \mathbb{C}^{\mathsf{Q}_0}$ by

$$\begin{aligned}
\lambda_{[i,j]} &:= -\xi_1^{[i,j]} && \text{for } i\in I_{\mathrm{irr}}\backslash\{0\},\ j=1,\dots,m^{(i)},\\
\lambda_{[0,j]} &:= -\xi^{[0,j]} - \sum_{i\in I_{\mathrm{reg}}} \xi_1^{[i,1]} && \text{for } j=1,\dots,m^{(0)},\\
\lambda_{[i,j,k]} &:= \xi_k^{[i,j]} - \xi_{k+1}^{[i,j]} && \text{for } \begin{array}{l} i=0,\dots,p,\ j=1,\dots,m^{(i)},\\ k=1,\dots,e_{[i,j]}-1. \end{array}
\end{aligned}$$

Also define a sublattice of $\mathbb{Z}^{\mathsf{Q}_0}$,

$$\mathcal{L} = \left\{ \beta\in\mathbb{Z}^{\mathsf{Q}_0} \;\middle|\; \sum_{j=1}^{m^{(0)}} \beta_{[0,j]} = \sum_{j=1}^{m^{(i)}} \beta_{[i,j]} \text{ for all } i\in I_{\mathrm{irr}}\backslash\{0\} \right\}.$$

Set $\mathcal{L}^+ = \mathcal{L}\cap(\mathbb{Z}_{\ge 0})^{\mathsf{Q}_0}$.

2.2.1. $\mathfrak{M}(\mathbf{B})$ *and a quiver variety.* Now we shall give an identification of $\mathfrak{M}(\mathbf{B})$ with a subspace of the quiver variety $\mathfrak{M}_\lambda(\mathsf{Q},\alpha)$. Before seeing this, we introduce $\mathcal{L}$-irreducible representations in $\mu^{-1}(\lambda)$ which are defined by a weaker condition than the irreducibility.

Definition 2.1 ($\mathcal{L}$-irreducible). If $x\in\mu^{-1}(\lambda)$ has no nontrivial proper subrepresentation $\{0\}\neq y\subsetneq x$ in $\mu^{-1}(\lambda)$ with $\mathbf{dim}\, y\in\mathcal{L}$, then x is said to be *$\mathcal{L}$-irreducible.*

Then we have the following bijection from $\mathfrak{M}(\mathbf{B})$ onto a subset of the quiver variety $\mathfrak{M}_\lambda(\mathsf{Q},\alpha)$.

Theorem 2.2 (Theorem 5.14 in [14]). *There exists a bijection*

$$\Phi_{\mathbf{B}} \colon \mathfrak{M}(\mathbf{B}) \longrightarrow \mathfrak{M}_\lambda(\mathsf{Q}, \alpha)^{dif}$$

where

$$\mathfrak{M}_\lambda(\mathsf{Q}, \alpha)^{dif} :=$$

$$\left\{ x \in \mu^{-1}(\lambda) \;\middle|\; \begin{array}{c} x \text{ is } \mathcal{L}\text{-irreducible}, \\ \det\left(x_{\rho^{[0,j]}_{[i,j']}} \right)_{\substack{1 \le j \le m^{(0)} \\ 1 \le j' \le m^{(i)}}} \neq 0, i \in I_{irr} \backslash \{0\} \end{array} \right\} \Big/ \mathbf{G}.$$

As an analogy of Corollary 1.10 by Crawley-Boevey, we can determin a necessary and sufficient condition for $\mathfrak{M}(\mathbf{B}) \neq \emptyset$. Define a set $\Sigma_\lambda^{\mathrm{dif}}$ consists of $\beta \in \mathcal{L}^+$ satisfying

(1) β is a positive root of Q and $\beta \cdot \lambda = 0$,
(2) for any decomposition $\beta = \beta_1 + \cdots + \beta_r$ where $\beta_i \in \mathcal{L}^+$ are positive roots of Q satisfying $\beta_i \cdot \lambda = 0$, we have

$$p(\beta) > p(\beta_1) + \cdots + p(\beta_r).$$

Theorem 2.3 (Non-emptiness of moduli spaces. Theorem 0.9 in [14]). *The moduli space $\mathfrak{M}(\mathbf{B}) \neq \emptyset$ if and only if $\alpha \in \Sigma_\lambda^{dif}$.*

Let us recall the *spectral type* which is already appeared in Section 1.2 in [23]. Consider the inductive limit

$$\mathbb{Z}^\infty := \varinjlim \mathbb{Z}^n$$

defined by inclusions $\phi_{i,i+1} \colon \mathbb{Z}^i \ni (a_1, \ldots, a_i) \mapsto (a_1, \ldots, a_i, 0) \in \mathbb{Z}^{i+1}$ for $i = 1, 2, \ldots$.

Definition 2.4 (spectral type and index of rigidity). The spectral type of $\mathbf{B}$ is the pair

$$\left(\mathbf{m}_\alpha, \left(d_i(j,j')\right)_{\substack{i=0,\ldots,p \\ 1 \le j < j' \le m^{(i)}}} \right)$$

where $\mathbf{m}_\alpha = \left((m_{[i,j,1]}, \ldots, m_{[i,j,e_{[i,j]}]}) \right)_{\substack{0 \le i \le p \\ 1 \le j \le m^{(i)}}} \in \bigoplus_{i=0}^p \bigoplus_{j=1}^{m^{(i)}} \mathbb{Z}^\infty$ which satisfies $\sum_{j=1}^{m^{(0)}} \sum_{k=1}^{e_{[0,j]}} m_{[0,j,k]} = \cdots = \sum_{j=1}^{m^{(p)}} \sum_{k=1}^{e_{[p,j]}} m_{[p,j,k]}$ is defined by

$$m_{[i,j,k]} := \alpha_{[i,j,k-1]} - \alpha_{[i,j,k]}$$

where

$$\alpha_{[i,j,0]} = \begin{cases} \alpha_{[i,j]} & \text{if } i \in I_{\mathrm{irr}}, \\ \sum_{k=1}^{m^{(0)}} \alpha_{[0,k]} & \text{if } i \in I_{\mathrm{reg}} \end{cases}$$

and $\alpha_{[i,j,e_{[i,j]}]} = 0$. Sometimes we write $\mathbf{m}_\alpha = (\mathbf{m}_\alpha, d_i(j,j'))$ for short.

The *index of rigidity* of $\mathbf{m}_\alpha$ is defined by

$$\mathrm{idx}\mathbf{m} := 2q(\alpha).$$

For convenience we introduce the following notation for $\mathbf{m}$. The each number $d_i(j,j') + 1$ is expressed by the number of parentheses ( ) between the sequences $m_{[i,j,1]}, m_{[i,j,2]}, \ldots$ and $m_{[i,j',1]}, m_{[i,j',2]}, \ldots$. For instance, if

$$\mathbf{m}_\beta = \cdots m_{[i,j,1]} m_{[i,j,2]} \cdots m_{[i,j,l_{i,j}]})) ((m_{[i,j',1]} m_{[i,j',2]} \cdots,$$

then the double parenthesis $))\,(($ between $m_{[i,j,1]}\ldots$, and $m_{[i,j',1]}\ldots$ means $d_i(j,j')=1$.

For example, put $p=1$,

$$(m^{(0)},m^{(1)})=(2,3),\quad (e_{[0,1]},e_{[0,2]},e_{[1,1]},e_{[1,2]},e_{[1,3]})=(1,2,1,1,2),$$
$$(d_0(1,2),d_1(1,2),d_1(2,3),d_1(1,3))=(0,0,1,1).$$

Then $\mathbf{m}=((m_{[i,j,1]},\ldots,m_{[i,j,l_{i,j}]}))_{\substack{0\le i\le p\\ 1\le j\le k_i}}$ is written by

$$(m_{[0,1,1]})(m_{[0,2,1]}m_{[0,2,2]}),\ ((m_{[1,1,1]})(m_{[1,2,1]}))((m_{[1,3,1]}m_{[1,3,2]})).$$

2.3. **Integrable deformation.** Let us introduce *integrable admissible families* of connections following Boalch [3] and Yamakawa [41].

Let $\mathbb{T}$ be a contractible complex manifold and $a_i\colon \mathbb{T}\to\mathbb{P}^1\times\mathbb{T}$, $i=0,\ldots,p$, holomorphic sections of the fiber bundle $\pi\colon\mathbb{P}^1\times\mathbb{T}\to\mathbb{T}$. Moreover assume that

$$a_i(t)\neq a_j(t)\text{ if } i\neq j$$

in each fiber $\mathbb{P}^1_t:=\mathbb{P}^1\times\{t\}$. Moreover we fix a standard coordinate $z\colon\mathbb{P}^1_t\cong\mathbb{C}\cup\{\infty\}$ so that $a_0(s)=\infty$ and $d_{\mathbb{T}}z=0$ on the trivial bundle $\mathbb{P}^1\times\mathbb{T}\to\mathbb{T}$. Let us set

$$z_i\colon\mathbb{P}^1\times\mathbb{T}\to\mathbb{T};\quad (z,t)\mapsto\begin{cases}1/z & (i=0)\\ z-a_i(t) & (i\neq 0)\end{cases}$$

for $i=0,\ldots,p$. Let us consider a family $\mathbf{B}(t)=(B^{(i)}(t))_{i=0,\ldots,p}$ of collections of HTL normal forms of the forms

$$B^{(i)}(t)=$$
$$\operatorname{diag}\left(q_1^{(i)}(t,z_i^{-1})I_{n_1^{(i)}}+R_1^{(i)}(t)z_i^{-1},\ldots,q^{(i)}_{m^{(i)}}(t,z_i^{-1})I_{n^{(i)}_{m^{(i)}}}+R^{(i)}_{m^{(i)}}(t)z_i^{-1}\right).$$

Here all mappings $\mathbb{T}\ni t\mapsto q_j^{(i)}(t,z)\in\mathbb{C}[z]$ and $\mathbb{T}\ni t\mapsto R_j^{(i)}(t)\in M(n_j^{(i)},\mathbb{C})$ depend smoothly on $t\in\mathbb{T}$. Define $d_i(t;j,j'):=\deg_{\mathbb{C}[z]}(q_j^{(i)}(t,z)-\bar{q}_{j'}^{(i)}(t,z^{-1}))-2$. We say that $\mathbf{B}(t)$ is an *admissible family*[1] of the collections of HTL normal forms if $d_i(t;j,j')$ and $R_j^{(i)}(t)$ are independent of t for all $i=0,\ldots,p$ and $j,j'=1,\ldots,m^{(i)}$.

Let $(\mathbf{B}(t))_{t\in\mathbb{T}}$ be an admissible family of collections of HTL normal forms. Then as we saw in Remark refinvariance, we can find quiver Q, $\alpha\in\mathbb{Z}^{\mathsf{Q}_0}$ and $\lambda\in\mathbb{C}^{\mathsf{Q}_0}$ independently of $t\in\mathbb{T}$ such that we have isomorphisms

$$\Phi_{\mathbf{B}(t)}\colon\mathfrak{M}(\mathbf{B}(t))\xrightarrow{\sim}\mathfrak{M}_\lambda(\mathsf{Q},\alpha)^{\mathrm{dif}}$$

for all $t\in\mathbb{T}$. We further say that the admissible family $(\mathbf{B}(t))_{t\in\mathbb{T}}$ is *non-resonant* if eigenvalues of $R_j^{(i)}(t)$ never differ by any integer for each $i=0,\ldots,p$ and $j=1,\ldots,m^{(i)}$, which is equivalent to the condition,

$$\lambda_{[i,j,k]}\notin\mathbb{Z}\backslash\{0\}\text{ for all }[i,j,k]\in\mathsf{Q}_0^{\mathrm{leg}}.$$

[1]This is a little stronger condition than that in [41].

Definition 2.5 (admissible family). Then the family $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t\in\mathbb{T}}$ of meromorphic connections is called an *admissible family* with $(\mathbf{B}(t))_{t\in\mathbb{T}}$ if the followings are satisfied:

(1) the admissible family $(\mathbf{B}(t))_{t\in\mathbb{T}}$ is non-resonant.
(2) We have $(\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t) \in \mathfrak{M}(\mathbf{B}(t))$ for all $t \in \mathbb{T}$.
(3) For each $i = 0, \ldots, p$ and fixed $t \in \mathbb{T}$, let us write $\nabla_t = d - A_i(t, z_i)\, dz_i$, $A_i(t, z_i) \in M(n, \mathbb{C}((z_i)))$ near $z_i = 0$. Then there exists a holomorphic map $\widehat{g}_i \colon \mathbb{T} \to \mathrm{GL}(n, \mathbb{C}[\![z_i]\!])$ such that
$$A_i(t, z_i) = \widehat{g}_i(t)[B^{(i)}(t)].$$

As we see above, we can define the triple $(\mathsf{Q}, \lambda, \alpha)$ from $(\mathbf{B}(t))_{t\in\mathbb{T}}$. We call this triple the *spectral data* of the admissible family $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t\in\mathbb{T}}$ with $(\mathbf{B}(t))_{t\in\mathbb{T}}$. We call the number $2p(\alpha) = \dim(\mathfrak{M}_\lambda(\mathsf{Q}, \alpha)) = \dim(\mathfrak{M}(\mathbf{B}(t)))$, the *dimension* of the admissible family.

Definition 2.6 (integrable family). Let $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t\in\mathbb{T}}$ be an admissible family with $(\mathbf{B}(t))_{t\in\mathbb{T}}$. If there exists a flat meromorphic connection $\widehat{\nabla}$ on $\mathcal{O}^n_{\mathbb{P}^1\times\mathbb{T}}$ with poles on $\bigcup_{i=0}^p a_i(\mathbb{T})$ such that $\widehat{\nabla}|_{\mathbb{P}^1_t} = \nabla_t$, then we say that the family $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t\in\mathbb{T}}$ is *integrable*. In this case such $(\mathcal{O}^n_{\mathbb{P}^1\times\mathbb{T}}, \widehat{\nabla})$ is called a *flat extension* of $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t\in\mathbb{T}}$.

3. Middle convolutions, Weyl groups and integrable deformations

In this section, we see the relationship between middle convolutions and Weyl groups of quivers and give a classification of their symmetries in certain lower dimensional cases. And we see the symmetries of integrable families as an application.

3.1. Middle convolution and reflection functor. Let us take $(\mathcal{O}^n, \nabla) \in \mathfrak{M}(\mathbf{B})$ and write
$$\nabla = d - \left(\sum_{i=1}^{p}\sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{(z-a_i)^\nu} - \sum_{2\le\nu\le k_0} A_\nu^{(0)} z^{\nu-2}\right) dz.$$
Set
$$\mathbf{A} = (\sum_{j=1}^{k_i} A_j^{(i)} z^{-j})_{0\le i\le p} \in \prod_{i=0}^{p} \mathcal{O}_{B^{(i)}}$$
where $A_1^{(0)} := -\sum_{i=1}^p A_1^{(i)}$. Set
$$\mathcal{J}_i := \{[i,j] \mid j = 1, \ldots, m^{(i)}\} \quad \text{for } i = 0, \ldots, p$$
and
$$\mathcal{J} := \prod_{i=0}^{p} \mathcal{J}_i.$$
Then we can define an operation called *middle convolution* for meromorphic connections on trivial bundles over $\mathbb{P}^1$, see [10], [9],[37], [1],[22],[35],and [40].

Thus form the connection ∇ we can define a new connection $\mathrm{mc}_{\mathbf{i}}(\nabla)$ on a trivial bundle $\mathcal{O}^{n'}$ over $\mathbb{P}^1$ satisfying the following properties.

Suppose we can choose $\mathbf{i} \in \mathcal{J}$ so that $\xi_{\mathbf{i}} \neq 0$.

(1) If ∇ is stable, then $\mathrm{mc}_{\mathbf{i}}(\nabla)$ is stable.
(2) If ∇ is stable,

$$\mathrm{mc}_{\mathbf{i}} \circ \mathrm{mc}_{\mathbf{i}}(\mathbf{A}) \sim \mathbf{A},$$

i.e., there exists $g \in \mathrm{GL}(n, \mathbb{C})$ such that

$$\mathrm{mc}_{\mathbf{i}} \circ \mathrm{mc}_{\mathbf{i}}(\mathbf{A}) = g\mathbf{A}g^{-1} := (gA_i(z^{-1})g^{-1})_{0 \le i \le p}.$$

3.2. **Middle convolutions on representations of a quiver.** We shall define an analogy of the reflection functors for the subspace $\mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{\mathrm{dif}} \subset \mathfrak{M}_\lambda(\mathrm{Q}, \alpha)$ by using middle convolutions. For $\mathbf{i} = ([i, j_i])_{0 \le i \le p} \in \mathcal{J}$, let us define $\epsilon_{\mathbf{i}} \in \mathbb{Z}^{\mathrm{Q}_0}$ by

$$(\epsilon_{\mathbf{i}})_a := \begin{cases} 1 & \text{if } a = [i, j_i],\ i \in I_{\mathrm{irr}}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $\epsilon_{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{J}$ are positive real roots of Q. Let us define

$$s_{\mathbf{i}}(\beta) := \beta - (\beta, \epsilon_{\mathbf{i}})\epsilon_{\mathbf{i}}$$

for $\mathbf{i} \in \mathcal{J}$ and $\beta \in \mathbb{Z}^{\mathrm{Q}_0}$. Also define $r_{\mathbf{i}}(\mu)$ for $\mu \in \mathbb{C}^{\mathrm{Q}_0}$ by

$$r_{\mathbf{i}}(\mu)_{[i,j]} := \begin{cases} \mu_{[i,j]} & \text{if } [i,j] \neq [0, j_0], \\ \mu_{[0,j_0]} - 2\mu_{\mathbf{i}} & \text{if } [i,j] = [0, j_0], \end{cases}$$

$$r_{\mathbf{i}}(\mu)_{[i,j,k]} := \begin{cases} \mu_{[i,j,k]} & \text{if } [i,j,k] \neq [i, j_i, 1], \\ \mu_{[i,j_i,1]} + \mu_{\mathbf{i}} & \text{if } [i,j,k] = [i, j_i, 1]. \end{cases}$$

Then we can see that the middle convolutions induce the following operations on quiver varieties.

Theorem 3.1. *Let us consider $\mathfrak{M}(\mathbf{B}) \neq \emptyset$ and the corresponding quiver variety $\mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{dif}$ under the bijection in Theorem 2.2. Suppose that we can take $\mathbf{i} = ([i, j_i]) \in \mathcal{J}$ so that $\lambda_{\mathbf{i}} := \sum_{i \in I_{\mathrm{irr}}} \lambda_{[i,j_i]} = -\xi_{\mathbf{i}} \neq 0$. Then there exists a bijection*

$$s_{\mathbf{i}} \colon \mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{dif} \longrightarrow \mathfrak{M}_{r_{\mathbf{i}}(\lambda')}(\mathrm{Q}, s_{\mathbf{i}}(\alpha))^{dif}$$

3.3. **The lattice $\mathcal{L}$ as a Kac-Moody root lattice.** As we saw in Theorem 2.3, if $\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{\mathrm{dif}} \neq \emptyset$, then α must be in $\mathcal{L} \cap \Delta$ where Δ is the set of roots in $\mathbb{Z}^{\mathrm{Q}_0}$. This inclines us to see $\mathcal{L} \cap \Delta$ as an analogy of the set of roots of the lattice $\mathcal{L}$ which may not be a true Kac-Moody root lattice. It can be checked that $\mathcal{L}$ is generated by $\{\epsilon_a \mid a \in \mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}}\}$ over $\mathbb{Z}$ and $W^{\mathrm{mc}} = \langle s_a \mid a \in \mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}}\rangle$ acts on $\mathcal{L}$. This may lead us to believe that $\mathcal{L}$ can be seen as a root lattice with the set of simple roots $\{\epsilon_a \mid a \in \mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}}\}$ and the Weyl group W^{mc}. However elements in $\{\epsilon_a \mid a \in \mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}}\}$ are not independent over $\mathbb{Z}$ in general. Thus we shall introduce a new lattice $\widehat{\mathcal{L}}$ of which $\mathcal{L}$ can be seen as a quotient. Let us note that

(1) $$(\epsilon_{\mathbf{i}}, \epsilon_{\mathbf{i}'}) = 2 - \sum_{\substack{0\le i\le p \\ j_i\neq j_i'}} (d_i(j_i, j_i') + 2),$$

(2) $$(\epsilon_{\mathbf{i}}, \epsilon_{[i,j,k]}) = \begin{cases} -1 & \text{if } j = j_i \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases}$$

(3) $$(\epsilon_{[i,j,k]}, \epsilon_{[i',j',k']}) = \begin{cases} 2 & \text{if } [i,j,k] = [i',j',k'], \\ -1 & \text{if } (i,j) = (i',j') \text{ and } |k-k'| = 1, \\ 0 & \text{otherwise} \end{cases}$$

for $\mathbf{i}, \mathbf{i}' \in \mathcal{J}$ and $[i,j,k], [i',j',k'] \in \mathrm{Q}_0^{\mathrm{leg}}$. Thus we consider a new lattice $\widehat{\mathcal{L}}$ generated by the set of indeterminate

$$\mathcal{C} = \left\{ c_a \mid a \in \mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}} \right\},$$

and define a symmetric bilinear form $(\,,\,)$ on $\widehat{\mathcal{L}}$ in accordance with equations (1), (2) and (3). Then $\widehat{\mathcal{L}}$ becomes a symmetric Kac-Moody root lattice and we have a projection

$$\Xi \colon \widehat{\mathcal{L}} \longrightarrow \mathcal{L}$$

where for $\gamma = \sum_{c\in\mathcal{C}} \gamma_c c \in \widehat{\mathcal{L}}$, the image $\Xi(\gamma) = (\beta_a)_{a\in\mathrm{Q}_0}$ is given by

$$\beta_{[i,j]} = \sum_{\{\mathbf{i}=([i,j_i])\in\mathcal{J} | j_i = j\}} \gamma_{c_{\mathbf{i}}},$$
$$\beta_{[i,j,k]} = \gamma_{c_{[i,j,k]}}.$$

Then Theorem 3.6 in [13] shows that Ξ maps the Weyl group of $\widehat{\mathcal{L}}$ to W^{mc}. Namely we can say that $\widehat{\mathcal{L}}$ is a "lift" of $\mathcal{L}$ to a Kac-Moody root lattice with the Weyl group W^{mc}.

The kernel of Ξ is a big space in general. Thus if we consider the inverse image of an element $\beta \in \mathcal{L}$, it is convenient to restrict Ξ to some smaller space as follows. Fix $\beta \in \mathcal{L}$ and set $\mathcal{J}_\beta := \{([i,j_i]) \in \mathcal{J} \mid \beta_{[i,j_i]} \neq 0 \text{ for all } i \in I_{\mathrm{irr}}\}$ and $(\mathrm{Q}_0^{\mathrm{leg}})_\beta := \mathrm{Q}_0^{\mathrm{leg}} \cap \mathrm{supp}(\beta)$. Then define

$$(\mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}})_\beta := \mathcal{J}_\beta \cup (\mathrm{Q}_0^{\mathrm{leg}})_\beta$$

and a sublattice and subgroup

$$\widehat{\mathcal{L}}_\beta := \sum_{\{a\in(\mathcal{J}\cup\mathrm{Q}_0^{\mathrm{leg}})_\beta\}} \mathbb{Z}c_a,$$
$$W_\beta^{\mathrm{mc}} := \langle s_a \mid a \in (\mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}})_\beta \rangle.$$

Denote the set of all positive elements in $\widehat{\mathcal{L}}_\beta$ by $\widehat{\mathcal{L}}_\beta^+$. We write the restriction of Ξ on $\widehat{\mathcal{L}}_\beta$ by Ξ_β.

3.3.1. *A classification of spectral types.* Let us define an analogue of fundamental set of the root lattice $\mathbb{Z}^{\mathrm{Q}_0}$,

$$\tilde{F} := \left\{ \beta \in \mathcal{L}^{+}\backslash\{0\} \;\middle|\; \begin{array}{c} (\beta, \epsilon_a) \leq 0 \text{ for all } a \in \mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}} \\ \text{support of } \beta \text{ is connected} \end{array} \right\}$$

called *$\mathcal{L}$-fundamental set.* Then we can see that $\tilde{F}$ can be seen as a fundamental domain under the action of the group W^{mc}. Namely, we can show that quiver varieties with imaginary roots as dimension vectors can be reduced to $\mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{\mathrm{dif}}$ with $\alpha \in \tilde{F}$ by the action of W^{mc}.

Thus we shall consider a classification of elements in the set $\tilde{F}$. First let us introduce the shape of $\beta \in \mathcal{L}$.

Definition 3.2 (shape). Fix a Kac-Moody root lattice $L = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\alpha = \sum_{i \in I} m_i \alpha_i \in L$. For the Dynkin diagram of the support of α, we attach each coefficient m_i of α to the vertex corresponding to α_i, then we obtain the diagram with the coefficients, which we call the *shape* of α.

For example, if $\alpha = m_1\alpha_{i_1} + m_2\alpha_{i_2} + m_3\alpha_{i_3} \in L$ with the diagram of the support $\underset{\alpha_{i_1}}{\bigcirc}\!-\!\underset{\alpha_{i_2}}{\bigcirc}\!-\!\underset{\alpha_{i_3}}{\bigcirc}$, the diagram with coefficients is $\underset{\alpha_{i_1}}{\overset{m_1}{\bigcirc}}\!-\!\underset{\alpha_{i_2}}{\overset{m_2}{\bigcirc}}\!-\!\underset{\alpha_{i_3}}{\overset{m_3}{\bigcirc}}$.

By using this we define shapes of elements in $\mathcal{L}$ as follows.

Definition 3.3. For $\beta \in \mathcal{L}$, the *shape* of β is the set of shapes of elements in $\Xi_\beta^{-1}(\beta) \subset \widehat{\mathcal{L}}_\beta$.

We say that $\beta \in \mathbb{Z}_{\geq 0}^{\mathrm{Q}_0}$ is *reduced* if it never happens that there exists $i \in \{1, \ldots, p\}$ such that $\#\{j \mid \beta_{[i,j]} \neq 0\} = 1$ and $e_{[i,j_i]} = 1$ where $j_i \in \{j \mid \beta_{[i,j]} \neq 0\}$. Let us consider the set of all nonempty moduli spaces $\mathfrak{M}(\mathbf{B})$. Set

$$\mathrm{Ht}^{(n)} := \left\{ (B_i) \in \bigoplus^{\infty} M(n, \mathbb{C}[z^{-1}]) \;\middle|\; \text{all } B_i \text{ are HTL normal forms} \right\}$$

$$\mathrm{Ht} := \bigcup_{n=1}^{\infty} \mathrm{Ht}^{(n)}.$$

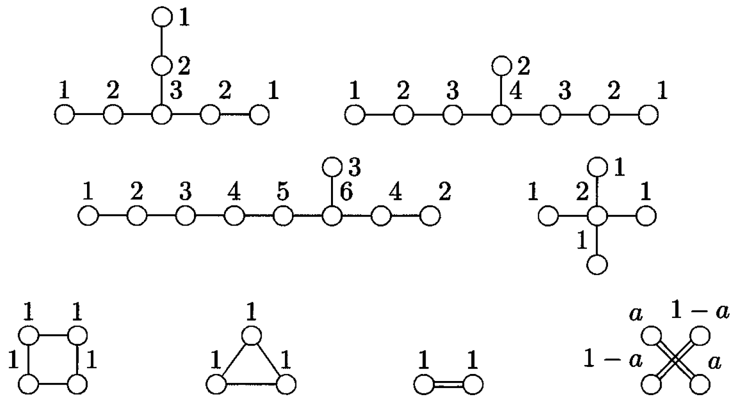
Definition 3.4 (fundamental spectral type). Let $\mathbf{m}$ be a spectral type. We say that $\mathbf{m}$ is *effective* if there exists $\mathbf{B} \in \mathrm{Ht}$ such that $\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{\mathrm{dif}} \neq \emptyset$ and $\mathbf{m} = \mathbf{m}_\alpha$. A spectral type $\mathbf{m} = \mathbf{m}_\alpha$ is said to be *basic* if $\alpha \in \tilde{F}$. Also we say that $\mathbf{m}$ is *reduced* if α is reduced. We say that $\mathbf{m}$ is *fundamental* if $\mathbf{m}$ is effective, basic and reduced. By the *shape* of $\mathbf{m}$, we mean the shape of α.

Then we can show the following finiteness of basic spectral types.

Theorem 3.5 (Theorem 8 in [16]). *Let us fix an integer $q \in 2\mathbb{Z}_{\leq 0}$. Then there exist only finite number of fundamental spectral types $\mathbf{m}$ satisfying* $\mathrm{idx}\mathbf{m} = q$.

Let us see the cases $q = 0$ and -2 for example. The first case is $q = 0$.

Theorem 3.6 (Theorem 9 in [16]). *Shapes of fundamental spectral types* $\mathbf{m}$ *satisfying* $\operatorname{idx}\mathbf{m} = 0$ *are one of the following.*



(1)(1), 11, 11 ((1))((1)), 11 (((1)))(((1))) ($a \in \mathbb{Z}$)

((1)(1))((1)(1)) ((1))((1))((1)) (1)(1), (1)(1)

We simply write sets $\{x_a \mid a \in \mathbb{Z}\}$ *and* $\{x\}$ *by* x_a $(a \in \mathbb{Z})$ *and* x*, respectively. For the first* 4 *star shaped graphs, corresponding spectral types are given in Remark 3.7 below.*

If $\mathfrak{M}_\lambda(\mathsf{Q},\alpha)^{\mathrm{dif}} \neq \emptyset$ and $\alpha \in \tilde{F}$ with $q(\alpha) = 0$, then by the above list of shapes of α, we can check that α is invariant under W^{mc}_α, i.e., $w(\alpha) = \alpha$ for any $w \in W^{\mathrm{mc}}_\alpha$. Then

$$s_a \colon \mathfrak{M}_\lambda(\mathsf{Q},\alpha)^{\mathrm{dif}} \longrightarrow \mathfrak{M}_{r_a(\lambda)}(\mathsf{Q},\alpha)^{\mathrm{dif}}$$

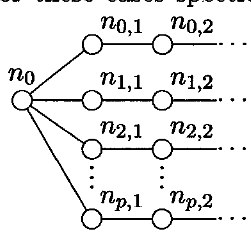
for each $a \in (\mathcal{J} \cup \mathsf{Q}_0^{\mathrm{leg}})_\alpha$ defines a W^{mc}_α-action on the parameter space

$$r_a \colon \begin{array}{ccc} \sum_{a \in (\mathcal{J} \cup \mathsf{Q}_0^{\mathrm{leg}})_\alpha} \mathbb{C}c_a & \longrightarrow & \sum_{a \in (\mathcal{J} \cup \mathsf{Q}_0^{\mathrm{leg}})_\alpha} \mathbb{C}c_a \\ \lambda & \longmapsto & r_a(\lambda) \end{array},$$

see also Proposition 3.7 in [13]. Here if $\lambda_a = 0$, i.e., s_a on $\mathfrak{M}_\lambda(\mathsf{Q},\alpha)^{\mathrm{dif}}$ is not well-defined, we formally set $s_a = \mathrm{id}$ and $r_a = \mathrm{id}$. By the above theorem, W^{mc}_α is isomorphic to one of the Weyl groups of the following types,

$$E_8^{(1)},\ E_7^{(1)},\ E_6^{(1)},\ D_4^{(1)},\ A_3^{(1)},\ A_2^{(1)},\ A_1^{(1)},\ A_1^{(1)} \times A_1^{(1)}.$$

Remark 3.7. In the above list of shapes, we omit the spectral types for star-shaped diagrams. For these cases spectral types are obtained as follows. Consider a shape



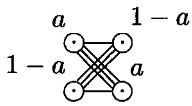
and put $m_{(i,1)} := n_0 - n_{i,1}$, $m_{(i,j+1)} := n_{i,j} - n_{i,j+1}$, $m_{(i,0)} := \sum_{\substack{0 \le k \le p \\ k \neq i}} n_{k,1} - n_0$ and $m_{(0)} := \sum_{i=0}^{p} n_{i,1} -$

n_0. Then the shape corresponds to the following 5 types.

$$\begin{aligned}
&m_{(0,1)}m_{(0,2)}\cdots,\ m_{(1,1)}m_{(1,2)}\cdots,\ \ldots,\ m_{(p,1)}m_{(p,2)}\cdots,\\
&m_{(0)}n_0,\ (m_{(0,2)}m_{(0,3)}\cdots)\cdots(m_{(p,2)}m_{(p,3)}\cdots),\\
&m_{(i,0)}m_{(i,1)}\cdots,\ (m_{(0,2)}m_{(0,3)}\cdots)\cdots(m_{(i-1,2)}\cdots)(m_{(i+1,2)}\cdots)\cdots,\\
&((m_{(i,1)}m_{(i,2)}\cdots))((m_{(0,2)}m_{(0,3)}\cdots)\cdots(m_{(i-2,2)}\cdots)(m_{(i+1,2)}\cdots)\cdots),\\
&((n_0))((m_{(0,2)}m_{(0,3)}\cdots)\cdots(m_{(p,2)}m_{(p,3)}\cdots)).
\end{aligned}$$

Next let us see the case $q = -2$.

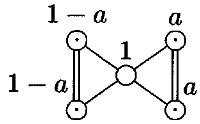
Theorem 3.8 (Theorem 10 in [16]). *Shapes of fundamental spectral types* $\mathbf{m}$ *satisfying* $\operatorname{idx}\mathbf{m} = -2$ *are one of the following.*



$(a \in \mathbb{Z})$

$((1))((1)), (1)(1)$

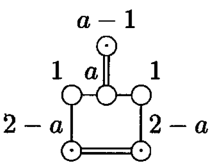
$W^{inv} = \emptyset$



$(a \in \mathbb{Z})$

$(1)(1), (1)(1), 11$

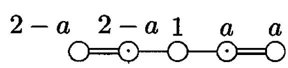
$W^{inv} = A_1$



$(a \in \mathbb{Z})$

$(1)(11), (1)(11)$

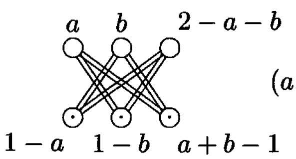
$W^{inv} = A_3$



$(a \in \mathbb{Z})$

$(2)(2), (2)(11)$

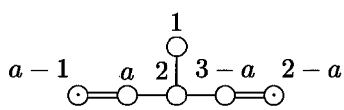
$W^{inv} = A_1 \times A_1 \times A_1$



$(a, b \in \mathbb{Z})$

$(1)(1)(1), (2)(1)$

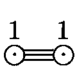
$W^{inv} = A_1 \times A_1 \times A_1$



$(a \in \mathbb{Z})$

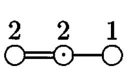
$(2)(2), (1)(111)$

$W^{inv} = D_4$



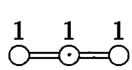
$((((1))))((((1))))$

$W^{inv} = \emptyset$



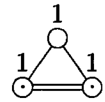
$(((2)))(((11)))$

$W^{inv} = A_1 \times A_1$



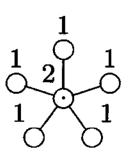
$(((1)(1)))(((1)))$

$W^{inv} = A_1 \times A_1$

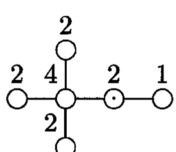


$(((1)))(((1))), 11$

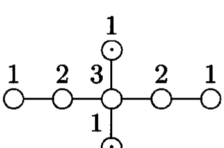
$W^{inv} = A_1$



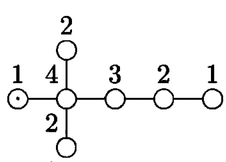
$W^{inv} = (A_1)^5$



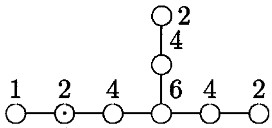
$W^{inv} = D_4 \times A_1$



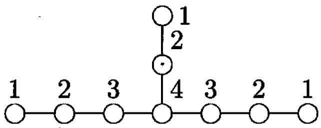
$W^{inv} = A_5$



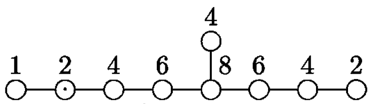
$W^{inv} = D_6$



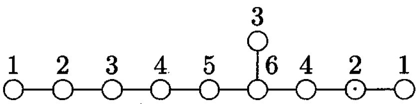
$W^{inv} = E_6 \times A_1$



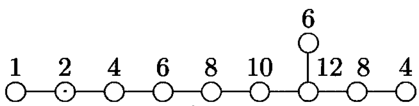
$W^{inv} = A_7 \times A_1$



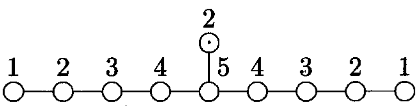
$W^{inv} = E_7 \times A_1$



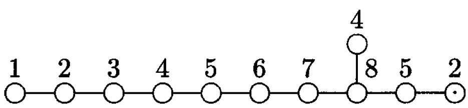
$W^{inv} = D_8 \times A_1$



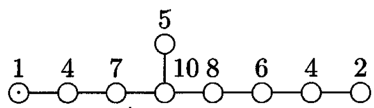
$W^{inv} = E_8 \times A_1$



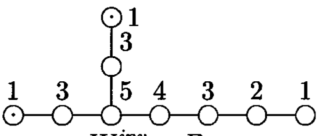
$W^{inv} = A_9$



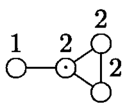
$W^{inv} = D_{10}$



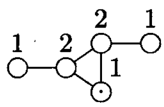
$W^{inv} = E_8$



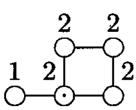
$W^{inv} = D_7$



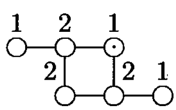
((2))((2))((11))
((2))((11)), 22
((2))((2)), 211
$W^{inv} = A_2 \times A_1$



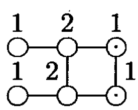
((11))((11))((1))
((11))((1)), 111
((11))((11)), 31
$W^{inv} = A_4$



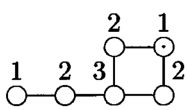
((2)(2))((2)(11))
(2)(2), 22, 211
(2)(11), 22, 22
$W^{inv} = A_3 \times A_1$



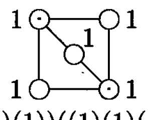
((11)(11))((2)(1))
(11)(11), 22, 31
(2)(1), 111, 111
$W^{inv} = A_5$



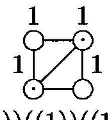
((11)(1))((11)(1))
((11)(1)), 21, 111
$W^{inv} = A_4$



((1)(111))((2)(2))
(1)(111), 22, 22
(2)(2), 31, 1111
$W^{inv} = D_5$



((1)(1))((1)(1)(1))
(1)(1), 11, 11, 11
(1)(1)(1), 21, 21
$W^{inv} = (A_1)^3$



((1))((1))((1)(1))
((1))((1)), 11, 11
((1)(1))((1)), 21
$W^{inv} = A_1 \times A_1$

Here we simply denote the sets $\{x_a \mid a \in \mathbb{Z}\}$ and $\{y\}$ by x_a $(a \in \mathbb{Z})$ and y, respectively. For the spectral types of the star shaped graphs, see Remark

3.7. Here for fundamental spectral types $\mathbf{m} = \mathbf{m}_\alpha$

$$W^{inv} := \langle s_a \mid s_a(\alpha) = \alpha, a \in (\mathcal{J} \cup \mathrm{Q}_0^{leg})_\beta \rangle \subset W_\alpha^{\mathrm{mc}}.$$

Plain circles in the Dynkin diagrams correspond to simple roots c_a *such that* $s_a(\beta) = \beta$ *and dotted circles correspond to* $s_a(\beta) \neq \beta$.

As well as the case $q = 0$, in the case of $q = -2$, $\mathfrak{M}_\lambda(\mathrm{Q}, \alpha)^{\mathrm{dif}} \neq \emptyset$ with $\alpha \in \tilde{F}$ such that $q(\alpha) = -2$ has a W^{inv}-action on the parameter space

$$r_a \colon \begin{array}{ccc} \sum_{a \in (\mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}})_\alpha} \mathbb{C} c_a & \longrightarrow & \sum_{a \in (\mathcal{J} \cup \mathrm{Q}_0^{\mathrm{leg}})_\alpha} \mathbb{C} c_a \\ \lambda & \longmapsto & r_a(\lambda) \end{array}.$$

3.4. **Integrable deformations and middle convolutions.** The theorem below connects the W^{mc}-action on $\mathfrak{M}(\mathbf{B})$ and on integrable deformations. The following theorem is obtained by Haraoka-Filipuk in [11] for Fuchsain cases, Boalch in [5] for simply-laced Q with $I_{\mathrm{irr}} = \{0\}$ and Yamakawa in [41] for general Q.

Theorem 3.9 (Yamakawa. Corollary 3.17 in [41]. cf. Haraoka-Filipuk [11] and Boalch [5]). *Let* $(\mathbf{B}(t))_{t \in \mathbb{T}}$ *be an non-resonant admissible family of collections of HTL normal forms which satisfies that that* $(B^{(0)}(t))_{irr} \equiv 0$ *and* $\mathrm{pr}_{res}(B^{(0)}(t))$ *is invertible. Let* $\left((\mathcal{O}^n_{\mathbb{P}^1_t} \nabla_t)\right)_{t \in \mathbb{T}}$ *be an admissible integrable family with* $(\mathbf{B}(t))_{t \in \mathbb{T}}$ *and the spectral data* $(\mathrm{Q}, \lambda, \alpha)$. *Then for each* $\mathbf{i} \in \mathcal{J}$, *there exists an admissible integrable deformation* $\left((\mathcal{O}^n_{\mathbb{P}^1_t} \nabla^{\mathbf{i}}_t)\right)_{t \in \mathbb{T}}$ *with the spectral data* $(\mathrm{Q}, r_{\mathbf{i}}(\lambda), s_{\mathbf{i}}(\alpha))$ *such that* $\nabla^{\mathbf{i}}_t \cong \mathrm{mc}_{\mathbf{i}}(\nabla_t)$ *for all* $t \in \mathbb{T}$.

Definition 3.10. We say that an admissible integral family $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t \in \mathbb{T}}$ with $(\mathrm{Q}, \lambda, \alpha)$ is *fundamental* when $\alpha \in \tilde{F} \cap \Sigma_\lambda^{\mathrm{dif}}$ and α is reduced.

Then Theorem refreduction and Theorem 3.9 show the following.

Theorem 3.11. *Let* $(\mathbf{B}(t))_{t \in \mathbb{T}}$ *be as in Theorem 3.9. Let* $((\mathcal{O}^n, \nabla_t))_{t \in \mathbb{T}}$ *be an admissible integrable family with* $(\mathbf{B}(t))_{t \in \mathbb{T}}$ *and the spectral data* $(\mathrm{Q}, \lambda, \alpha)$ *where* $\alpha \in \Sigma_\lambda^{dif}$ *and* $q(\alpha) \leq 0$. *Suppose that* λ *is generic (for the precise condition, see* [15]*). Then* $\left((\mathcal{O}_{\mathbb{P}^1_t}, \nabla_t)\right)_{t \in \mathbb{T}}$ *can be reduced to a fundamental admissible integral deformation by a finite iteration of middle convolutions and additions.*

For an admissible integrable family $\left((\mathcal{O}^n_{\mathbb{P}^1_t}, \nabla_t)\right)_{t \in \mathbb{T}}$ with a spectral data $(\mathrm{Q}, \lambda, \alpha)$, we call $\mathbf{m}_\alpha$ the *spectral type* and also λ the *spectral parameter.*

Theorem 3.12. *Let us fix an integer* $d \in 2\mathbb{Z}_{>0}$ *There exists only finite spectral types of fundamental admissible integrable deformations of dimension* d.

Proof. This directly follows from Theorems 3.5 and 3.9. □

We have the classification of spectral types of admissible deformations of dimension $d = 2$ and 4.

Theorem 3.13. *Spectral types of fundamental admissible integrable deformations of dimension* $d = 2, 4$ *are listed in Theorem 3.6 (resp. Theorem 3.8) for* $d = 2$ *(resp.* $d = 4$*). Moreover generic spectral parameters have* W^{mc}*-actions (resp.* W_{inv}*-action) for* $d = 2$ *(resp.* $d = 4$*).*

In [23], Kawakami, Nakamura and Sakai considered isomonodromic deformations of linear differential equations which obtained by the confluent process from Fuchsian differential equations with 4 accessory parameters classified by Oshima in [31]. And they gave explicit Hamiltonian equations of the isomonodromic deformations after Sakai's computation in the Fuchsian cases (see [33]). Then under the above identification of spectral types, Theorem 3.8 shows that the list of spectral types appeared in their paper [23] is the complete list of fundamental spectral types of dimension 4.

Theorem 3.14. *Under the above identification of spectral types, if we exclude the spectral types corresponding to differential equations which have only 3 regular singular points and no other singularities, then the list of spectral types appeared in Section 1.3 of* [23] *is the complete list of spectral types of fundamental integrable deformations of dimension 4.*

Moreover Theorem 3.13 assures that integrable deformations considered in [23] have W^{inv}- symmetries listed in Theorem 3.8.

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